HOLOMORPHIC FUNCTIONS ON BUNDLES OVER ANNULI

DAN ZAFFRAN

ABSTRACT. We consider a family $\{E_m(D,M)\}$ of holomorphic bundles constructed as follows: to any given $M \in GL_n(\mathbb{Z})$, we associate a "multiplicative automorphism" φ of $(\mathbb{C}^*)^n$. Now let $D \subseteq (\mathbb{C}^*)^n$ be a φ-invariant Stein Reinhardt domain. Then $E_m(D,M)$ is defined as the flat bundle over the annulus of modulus $m > 0$, with fiber D, and monodromy φ.

We show that the function theory on $E_m(D,M)$ depends nontrivially on the parameters m, M and D. Our main result is that

$$E_m(D,M) \text{ is Stein if and only if } m \log \rho(M) \leq 2\pi^2,$$

where $\rho(M)$ denotes the max of the spectral radii of M and M^{-1}.

As corollaries, we:
– obtain a classification result for Reinhardt domains in all dimensions;
– establish a similarity between two known counterexamples to a question of J.-P. Serre;
– suggest a potential reformulation of a disproved conjecture of Siu Y.-T.

Let D be a Stein manifold. We say that D belongs to $\mathcal{S}$ when: for any Stein manifold B and any locally trivial bundle $E \to B$, the manifold E is also Stein. A famous question of Serre can be formulated as: "Are all manifolds in $\mathcal{S}$?"

Skoda answered it in the negative, by proving that $\mathbb{C}^2 \notin \mathcal{S}$ (*cf.* [Sko]). Mok showed that any open Riemann surface belongs to $\mathcal{S}$ (*cf.* [Mok]).

Many bounded domains belong to $\mathcal{S}$: for any given bounded domain D, Diederich-Fornæss-Stehlé showed that if ∂D is smooth, then $D \in \mathcal{S}$ (*cf.* [Die-For] and [Steh]). Siu showed that if $b_1(D) = 0$ then $D \in \mathcal{S}$ (*cf.* [Siu]).

However, Cœuré and Lœb (*cf.* [Cœ-Lœ]) found a bounded domain D_{CL} not in $\mathcal{S}$. It happens that D_{CL} has the Reinhardt symmetry.

It is an open problem to characterize bounded domains of $\mathbb{C}^d$ not in $\mathcal{S}$ (*cf.* [Ch-Zh]). This classification problem is solved for all bounded *Reinhardt* domains with $d = 2$ in [Pfl-Zwo] and with $d = 2$ or 3 in [Oel-Zaf].

Here we study holomorphic functions on a family of bundles $\{E_m(D,M)\}$ over annuli, depending on a non necessarily bounded Reinhardt domain

---

Date: June 10, 2008.
zaffran@fudan.edu.cn
Fudan University, Shanghai.
Academia Sinica, Taipei.

$D \subset (\mathbb{C}^*)^d$, a matrix M and a number m. This family contains the non-Stein bundle of Cœuré and Lœb (*cf.* [Cœ-Lœ]). Moreover, roughly speaking, all known examples of bounded domains not in $\mathcal{S}$ appear as fibers in these bundles. Our main motivations are the problems of characterizing bounded (Reinhardt) domains not in $\mathcal{S}$, and also, when $D \notin \mathcal{S}$ is given, of characterizing the Steinness of a bundle with fiber D.

1. SETTING AND RESULTS

We work in the category of complex manifolds and holomorphic maps.

Let $d \geq 2$ and $M \in GL_d(\mathbb{Z})$. Then M gives an automorphism of $(\mathbb{C}^*)^d$ defined by $\boldsymbol{z} = (z_1, \ldots, z_d) \mapsto (w_1, \ldots, w_d)$ with $w_i = z_1^{M_{i1}} \ldots z_d^{M_{id}}$. This automorphism induces a $\mathbb{Z}$-action. We denote by $j.\boldsymbol{z}$ the action of $j \in \mathbb{Z}$ on $\boldsymbol{z} \in (\mathbb{C}^*)^d$. Fix $D \subset (\mathbb{C}^*)^d$ any Stein $\mathbb{Z}$-invariant Reinhardt domain.

Let $m > 0$. We realize the annulus of modulus m by

$$A_m = \left\{ w \in \mathbb{C} \;\middle|\; 1 < |w| < e^m \right\},$$

and its universal cover by the horizontal strip

$$S_m = \left\{ w \in \mathbb{C} \;\middle|\; \quad |Im\ w| < \frac{m}{4\pi} \right\}$$

endowed with the $\mathbb{Z}$-action generated by $w \mapsto w + 1$. Then $S_m/\mathbb{Z}$ is isomorphic to A_m.

There is a diagonal $\mathbb{Z}$-action on $S_m \times D$ given by $j.(w, \boldsymbol{z}) = (w + j, j.\boldsymbol{z})$, which is free and properly discontinuous. We consider the quotient manifold

$$E_m = E_m(D, M) = \frac{S_m \times D}{\mathbb{Z}}.$$

The projection map $S_m \times D \to D$ induces a locally trivial fibration

$$p : E_m \to A_m$$

with fiber D. We extend the notation to cover the cases of the two annuli of infinite modulus: $A_\infty = \left\{ w \in \mathbb{C} \;\middle|\; 1 < |w| < \infty \right\}$ (=the pointed disk) and $A_{2\infty} = \left\{ w \in \mathbb{C} \;\middle|\; 0 < |w| < \infty \right\}$ (= $\mathbb{C}^*$), with corresponding S_∞ (=$\mathbb{H}$), $S_{2\infty}$ (= $\mathbb{C}$) and bundles E_∞, $E_{2\infty}$.

We denote by $\rho(M)$ the max of the spectral radii of M and M^{-1}.

Main Theorem. $E_m(D, M)$ is Stein if and only if $m \log \rho(M) \leq 2\pi^2$.

Notice that we do not assume boundedness of D. By the methods of [Oel-Zaf], one can prove the Steinness when $\rho(M) = 1$. The results in that paper also imply the "only if" part for $d = 2$, and implicitly for $d = 3$.

The Steinness when $\rho(M) > 1$ and m is small enough is new even in small dimensions. Sections 2.1 and 2.2 are devoted respectively to the proofs of the "only if" and "if" parts of the Main Theorem. They both rely upon the existence or non-existence of fast decaying functions on S_m, and a Laurent series method already used in [Zaf].

1.1. **Case of a bounded** D**.** The "only if" part of the Main Theorem leads to a classification result that we now explain.

Let $D \subset (\mathbb{C}^*)^d$ be a bounded Reinhardt domain. Theorem 3 in [Oel-Zaf] yields: if no M with $\rho(M) > 1$ multiplicatively acts on D (see below), then $D \in \mathcal{S}$.

Let $m > 0$. Let $E' \to A_m$ be *any* fiber bundle over the annulus A_m with fiber D. It follows from [Roy] that E' is flat, so the monodromy $h \in Aut(D)$ characterizes $E' \to A_m$. By [Shim] we know that h is of the form

$$h(z_1, \dots, z_d) = (c_1\, z_1^{M_{11}} \!\cdot\!\cdot\!\cdot z_d^{M_{1d}}, \dots\dots, c_d\, z_1^{M_{d1}} \!\cdot\!\cdot\!\cdot z_d^{M_{dd}}),$$

with $M \in GL_d(\mathbb{Z})$ and $c_1, \dots, c_d \in \mathbb{C}^*$. When a given D admits such an automorphism, we say that M *multiplicatively acts* on D.

The bundle E' is very similar to $E_m(D, M)$. In fact it follows from the Main Theorem that E' is Stein if and only if $m \log \rho(M) \leq 2\pi^2$: If $Spec\ M = \{1\}$ (so $\rho(M) = 1$), then by Theorem 3 in [Oel-Zaf], E' is Stein. Up to applying (several times) Lemma 1.7 [Oel-Zaf], we can assume that $1 \notin Spec\ M$. Then, as shown in 3.3 [Oel-Zaf], there is an automorphism of $(\mathbb{C}^*)^d$ that conjugates h to $g : (z_1, \dots, z_d) \mapsto (z_1^{M_{11}} \!\cdot\!\cdot\!\cdot z_d^{M_{1d}}, \dots\dots, z_1^{M_{d1}} \!\cdot\!\cdot\!\cdot z_d^{M_{dd}})$. Therefore E' is isomorphic to $E_m(D, M)$ and we apply the Main Theorem.

The above implies in particular that as soon as M with $\rho(M) > 1$ multiplicatively acts on D, there exists a non-Stein bundle with fiber D over a Stein base (a thick enough annulus). So $D \notin \mathcal{S}$.

Summing up, the Main Theorem together with Theorem 3 in [Oel-Zaf] imply the

Theorem 1. *Let $D \subset (\mathbb{C}^*)^d$ be a bounded Stein Reinhardt domain, with $d \geq 2$. Then $D \in \mathcal{S}$ if and only if no M with $\rho(M) > 1$ multiplicatively acts on D.*

From the known cases of $d = 2$ or 3, we expect that the assumption that D do not intersect any coordinate hyperplane is only technical, and the result should hold unchanged for any bounded Reinhardt domain $D \subset \mathbb{C}^d$.

We use here a different method than in [Oel-Zaf] to obtain the higher-dimensional case, but the trade-off is that we lose some information. We don't know which matrices can appear for a bounded D not in $\mathcal{S}$, thus we don't know much about the geometry of such domains. However, the situation *does* become more complicated in dimension at least 4. For $d = 2, 3$, we showed in [Oel-Zaf] that a bounded D not in $\mathcal{S}$ can only be multiplicatively acted on by a real-diagonalizable matrix. This leads to a relatively simple geometric description of these domains. These facts do not extend to $d \geq 4$:

Proposition 1.1. *There exists a Stein bounded Reinhardt domain D in $(\mathbb{C}^*)^4$ that is multiplicatively acted on by a matrix $M \in GL_4^{\rho>1}(\mathbb{Z})$ with non-real spectrum. In particular, by Theorem 1, $D \notin \mathcal{S}$.*

Proof. Define

$$N := \begin{bmatrix} 0 & -2 & -7 & 9 \\ 0 & -10 & -20 & 29 \\ 0 & -13 & -31 & 43 \\ -1 & -11 & -36 & 47 \end{bmatrix} \in GL_4(\mathbb{Z}).$$

Then $Spec\ N = \{\alpha_1 = 6.23...,\ \alpha_2 = 0.27...,\ \omega = -0.25...+i0.73...,\ \bar{\omega}\}$. In particular, N is not similar to a block-diagonal matrix in $GL_2(\mathbb{Z}) \oplus GL_2(\mathbb{Z})$.

Direct computation shows that N admits real eigenvectors v_1 and v_2 associated to α_1 and α_2, and such that $v_i \in (\mathbb{R}^{<0})^4, i = 1 \dots 2$. Let $\{w', w''\}$ be a basis of the N-invariant real two-dimensional subspace of $\mathbb{R}^4$ corresponding to the complex eigenvalue ω.

Now take $u := [-14, -43, -62, -63]^t$. Direct computation shows that u decomposes as $u = a_1 v_1 + a_2 v_2 + a' w' + a'' w''$ with $a_1 > 0$ and $a_2 > 0$.

Denote the iterates $u_j := N^j u = (\alpha_1)^j a_1 v_1 + (\alpha_2)^j a_2 v_2 + w_j$. Then $\|w_j\|$ is of order $|\omega|^j$, but $|\alpha_2| < |\omega| < |\alpha_1|$ and $|\alpha_2| < 1 < |\alpha_1|$, so there exists $J > 0$ such that: for all $j \in \mathbb{Z}$, if $|j| \geq J - 4$ then $u_j \in (\mathbb{R}^{<0})^4$.

Define $A := \{u_{(2j+1)J+k} \mid j \in \mathbb{Z}, k = 0 \dots 4\}$, and Ω as the interior of the convex hull of A in $\mathbb{R}^4$. Then

- Ω is N^{2J}-invariant, because A is;
- Ω is contained in $(\mathbb{R}^{<0})^4$, because $|(2j+1)J + k| \geq J - 4$;
- Ω is not empty: By a rank computation (over $\mathbb{Z}$), one checks that $B := \{u_k \mid k = 0 \dots 4\}$ is not contained in any affine hyperplane of $\mathbb{R}^4$. Then, the same is true of A because A contains $\{u_{(2j+1)J+k} \mid j = 0, k = 0 \dots 4\} = N^J B$.

Then $D := \{\boldsymbol{z} \in (\mathbb{C}^*)^4 \mid (\log|z_1|, \dots, \log|z_4|) \in \Omega\}$ is a non-empty, bounded, Stein Reinhardt domain multiplicatively acted on by $M = N^{2J}$, and $\rho(M) = (\alpha_1)^{2J} > 1$. □

1.2. **Case of $D = (\mathbb{C}^*)^d$.** Take $M \in GL_2(\mathbb{Z})$ with $\rho(M) > 1$. Take $E_{2\infty} = E_{2\infty}\big((\mathbb{C}^*)^2, M\big)$, with fibration map $p : E_{2\infty} \to \mathbb{C}^*$. Then (*cf.* [Cœ-Lœ]) not only is $E_{2\infty}$ not Stein, but moreover $p^* : \mathcal{O}(\mathbb{C}^*) \to \mathcal{O}(E_{2\infty})$ is an isomorphism, i.e., all functions on $E_{2\infty}$ come from the base. In any dimension and over any annulus we prove the

Theorem 2. *Take $d \geq 2$, $M \in GL_d^{\rho>1}(\mathbb{Z})$ and m any positive number. Let $E_m = E_m\big((\mathbb{C}^*)^d, M\big)$. If the characteristic polynomial of M is irreducible over $\mathbb{Z}$ and $m \log \rho(M) > 2\pi^2$, then the fibration map $p : E_m \to A_m$ induces an isomorphism between functions on A_m and E_m. (Proof is in Sect. 2.3)*

So for any bundle E_m considered in this theorem, there is a critical value for m below which E_m is Stein, and above which the only functions come from the base. As opposed to this situation, recall that when D is a bounded domain and E_m is a bundle with fiber D over any Stein base, it is known that functions on E_m separate points (*cf.* [Siu]).

1.3. **Remark: Connection with the Schinzel-Zassenhaus problem.** From the "only if" part of the Main Theorem and Theorem 2, we obtain the

Corollary 1.2. *There exists a function $\mu : \mathbb{N} \to \mathbb{R}^{>0}$ such that for all $d > 0$ and $M \in GL_d^{\rho>1}(\mathbb{Z})$:*

(a) *for any Reinhardt domain D invariant by the multiplicative action of M, if $m > \mu(d)$ then $E_m(D, M)$ is not Stein;*
(b) *if $m > \mu(d)$ and the characteristic polynomial of M is irreducible, then any function on $E_m\big((\mathbb{C}^*)^d, M\big)$ is constant on fibers.*

Proof. For any polynomial P, denote by $\rho(P)$ the maximal modulus of its roots. Fix an integer $d \geq 1$. From the relations between the roots and coefficients of a polynomial, it follows that for any $a \in \mathbb{R}$, the set

$$\left\{ P \in \mathbb{Z}[X] \,\middle|\, P \text{ monic}, \ \deg P = d, \ \rho(P) \leq a \right\}$$

is finite. The corresponding set of roots is finite, so there exists $\mu'(d) > 0$ such that if P is any monic, integral polynomial of degree d, $\rho(P) > 1$ implies $\rho(P) \geq 1 + \mu'(d)$. We take $\mu'(d)$ minimal with that property.

Now let $M \in GL_d^{\rho>1}(\mathbb{Z})$ with characteristic polynomial P_M. Up to inverting M, we can assume that $\rho(P_M) = \rho(M)$. By assumption, $\rho(M) > 1$, so $\rho(M) \geq 1 + \mu'(d)$. Thus if we define $\mu(d) := \frac{2\pi^2}{\log\big(1+\mu'(d)\big)}$, the results follow from the Main Theorem and Theorem 2. $\square$

It is of independent interest to find estimates of $\mu'(d)$. Note that $\mu'(d) \leq \rho(X^d - 2) - 1 \sim \frac{\log 2}{d}$. In 1965, Schinzel and Zassenhaus asked in [S-Z] whether there exists $\gamma > 0$ independent of d such that $\mu'(d) \geq \frac{\gamma}{d}$.

Only partial results are known (see the survey [Smy]). In 1971, Smyth answered that question positively for the class of all non-reciprocal polynomials. One of the best general results, proved by Voutier in [Vou], is

$$\mu'(d) \geq \frac{1}{4d}\left(\frac{\log\log d}{\log d}\right)^3.$$

Each of these estimates yields an estimate on the function $\mu(d) = \frac{2\pi^2}{\log\big(1+\mu'(d)\big)}$.

On the other hand, if P is a monic, integral, degree d reciprocal polynomial, its companion matrix belongs to $GL_d(\mathbb{Z})$. Therefore, by the "if" part of the Main Theorem and Smyth's result, the Schinzel-Zassenhaus problem is equivalent to the existence of γ such that:

$$\text{If } m > \frac{2\pi^2}{\log(1+\frac{\gamma}{d})} \text{ then for all } M \in GL_d^{\rho>1}(\mathbb{Z}), \ E_m \text{ is not Stein.}$$

Note that asymptotically with respect to d, the condition is simply $m > \gamma' d$.

1.4. **Remark: Stein bundles over a non-Stein base.** Let $E \to B$ be a fiber bundle with fiber D. As D appears as a closed submanifold of E, it is necessary that D be Stein for E to be Stein. On the other hand, it can happen that B is not Stein but E is (e.g., $SL_2(\mathbb{C}) \to \mathbb{C}^2 - \{0\}$ with fiber $\mathbb{C}$). We refer to the articles by M. Abe for related results, and simply notice the following facts. Take $d = 2$ and $M \in GL_d^{\rho>1}(\mathbb{Z})$. Then for certain choices of a bounded D (e.g., $D = D_{CL}$), the manifold E_m admits *another* fibration $E_m \to \mathcal{C}$ with fiber S_m and base a non-Stein manifold $\mathcal{C}$ studied by Hirzebruch (*cf.* [Zaf]). Note that the pair $(S_m, w \mapsto w+1)$ is isomorphic to (Δ, ψ), where Δ is the unit disk in $\mathbb{C}$ and ψ some automorphism. Varying m corresponds to varying ψ. Then one can construct a continuous family $\{B_t\}_{t\geq 0}$ of disk bundles over $\mathcal{C}$ such that: B_0 is the trivial bundle (corresponding to ψ being the identity); by the Main Theorem, there exists $s > 0$ such that B_t is Stein if and only if $t \geq s$.

2. MAIN PROOFS.

2.1. **Proof of the "only if" part of the Main Theorem.** We want to show: If E_m is Stein then $m \log \rho(M) \leq 2\pi^2$.

For the sake of clarity, we denote vector quantities by bold characters. We denote by

$$\begin{aligned} q : S_m \times D &\to E_m \\ (w, \boldsymbol{z}) &\mapsto [w, \boldsymbol{z}] \end{aligned}$$

the quotient map that was used to define E_m. It induces an isomorphism between the spaces of scalar-valued functions $\mathcal{O}\big(E_m\big)$ and $\mathcal{O}^{\mathbb{Z}}\big(S_m \times (\mathbb{C}^*)^d\big)$.

For $f \in \mathcal{O}\big(E_m\big)$, we will use implicitly that isomorphism and write indifferently $f[w, \boldsymbol{z}]$ or $f(w, \boldsymbol{z})$.

We denote an element $\boldsymbol{k} \in \mathbb{Z}^d$ by a row vector $\boldsymbol{k} = (k_1, \ldots, k_d)$. For $\boldsymbol{z} \in (\mathbb{C}^*)^d$, we denote by $\boldsymbol{z}^{\boldsymbol{k}}$ the product $z_1^{k_1} \ldots z_d^{k_d} \in \mathbb{C}$.

Let $\Delta \subset S_m$ be a small disk centered at w_0. Then $\Delta \times D$ is a Reinhardt domain, so we can expand f there into a Laurent series:

$$\text{(1)} \qquad \text{for all } (w, \boldsymbol{z}) \in \Delta \times D, \quad f(w, \boldsymbol{z}) = \sum_{i \in \mathbb{N}, \boldsymbol{k} \in \mathbb{Z}^d} a_{i\boldsymbol{k}} (w - w_0)^i \boldsymbol{z}^{\boldsymbol{k}}.$$

Thanks to absolute convergence of such a series, we can write

$$\text{(2)} \qquad f(w, \boldsymbol{z}) = \sum_{\boldsymbol{k}} \Big(\sum_i a_{i\boldsymbol{k}} (w - w_0)^i \Big) \boldsymbol{z}^{\boldsymbol{k}} = \sum_{\boldsymbol{k}} g_{\boldsymbol{k}}(w) \boldsymbol{z}^{\boldsymbol{k}}.$$

Notice that by the uniqueness of Laurent expansions, for all $\boldsymbol{k}$, the function $g_{\boldsymbol{k}}$ so defined is actually a well-defined element of $\mathcal{O}(S_m)$, and (2) is valid on $S_m \times D$. Moreover, as (1) is locally absolutely uniformly converging with respect to $(w, \boldsymbol{z})$, so is the rightmost series in (2). This expansion of f is called a Hartogs-Laurent series.

For $j \in \mathbb{Z}$ and $\boldsymbol{k} \in \mathbb{Z}^d$, we denote $j.\boldsymbol{k} = \boldsymbol{k}M^j$. This defines a $\mathbb{Z}$-action on $\mathbb{Z}^d$ and moreover, for all j, $\boldsymbol{z}$ and $\boldsymbol{k}$,

$$(j.\boldsymbol{z})^{\boldsymbol{k}} = \boldsymbol{z}^{j.\boldsymbol{k}}. \tag{3}$$

As f is $\mathbb{Z}$-invariant, we know that for all j, w and $\boldsymbol{z}$, $f(w, \boldsymbol{z}) = f(w + j, j.\boldsymbol{z})$, so by (2) and (3),

$$\sum_{\boldsymbol{k}} g_{\boldsymbol{k}}(w)\boldsymbol{z}^{\boldsymbol{k}} = \sum_{\boldsymbol{k}} g_{\boldsymbol{k}}(w + j)\boldsymbol{z}^{j.\boldsymbol{k}}.$$

Uniqueness of the Hartogs-Laurent expansion implies that for all $j, \boldsymbol{k}$ and w, $g_{j.\boldsymbol{k}}(w) = g_{\boldsymbol{k}}(w + j)$.

Let $\mathcal{S}_0$ be the set of elements in $\mathbb{Z}^d$ with a finite $\mathbb{Z}$-orbit. If $\mathcal{S}_0 = \mathbb{Z}^d$, then $Spec\ M \subset S^1$, so $\rho(M) = 1$; thus the "only if" part of the theorem becomes trivially true. Hence we can assume that $\mathbb{Z}^d - \mathcal{S}_0$ is non-empty. The $\mathbb{Z}$-action restricted to $\mathbb{Z}^d - \mathcal{S}_0$ is free. Choosing an arbitrary section $\mathcal{S}$ of this free action, we can write

$$f(w, \boldsymbol{z}) = \sum_{\boldsymbol{k} \in \mathcal{S}_0} g_{\boldsymbol{k}}(w)\boldsymbol{z}^{\boldsymbol{k}} + \sum_{\boldsymbol{k} \in \mathcal{S}} \sum_{j \in \mathbb{Z}} g_{\boldsymbol{k}}(w + j)\boldsymbol{z}^{j.\boldsymbol{k}}.$$

Now for any fixed $\boldsymbol{k} \in \mathcal{S}$, the series

$$\sum_{j \in \mathbb{Z}} g_{\boldsymbol{k}}(w + j)\boldsymbol{z}^{j.\boldsymbol{k}}$$

is a subseries of (2) because $g_{\boldsymbol{k}}(w+j) = g_{j.\boldsymbol{k}}(w)$, hence it is locally absolutely uniformly converging. In particular, for all $\boldsymbol{z} \in D$ and $\boldsymbol{k} \in \mathcal{S}$

$$g_{\boldsymbol{k}}(w + j)\boldsymbol{z}^{j.\boldsymbol{k}} \xrightarrow{j \to \pm\infty} 0 \tag{4}$$

locally uniformly with respect to w.

Vocabulary. A subset $J \subset \mathbb{Z}$ is said to have *bounded gaps* not bigger than l when: for all $j \in J$, there exists $j' \in J - \{j\}$ such that $|j - j'| \leq l$.

The proof of the following proposition, which is a Phragmén-Lindelöf-type result (*cf.* [Tit] 5.65), is directly adapted from a proof that was kindly communicated to me by A. Baernstein and L. Kovalev.

Proposition 2.1. *Let* $\delta, \mu, h \in \mathbb{R}$ *such that* $\delta > 0$, $\mu > 0$ *and* $h > \frac{\pi}{\mu}$. *Let* $S = \left\{ w \in \mathbb{C} \,\middle|\, |Im\ w| < \frac{h}{2} \right\}$ *and* $g \in \mathcal{O}(S)$. *Assume that* w*-locally uniformly,*

$$g(w + j)\ e^{\delta e^{j\mu_+}} \xrightarrow[j \in J_+]{j \to +\infty} 0 \quad \text{and} \quad g(w + j)\ e^{\delta e^{|j|\mu_-}} \xrightarrow[j \in J_-]{j \to -\infty} 0,$$

where J_+,J_- *are infinite subsets of* $\mathbb{Z}$ *with bounded gaps, and* $\max(\mu_+, \mu_-) = \mu$. *Then* g *vanishes identically.*

Proof. Choose an h' such that $\frac{\pi}{\mu} < h' < h$. For $\alpha \in (0, +\infty]$, define

$$R_\alpha = \left\{ w \in \mathbb{C} \,\middle|\, |Re\ w| \leq \alpha\ ,\ |Im\ w| \leq h' \right\},$$

$R_\alpha^+ = R_\alpha \cap \{Re\ w \geq 0\}$ and $R_\alpha^- = R_\alpha \cap \{Re\ w \leq 0\}$.

Let l be a majorant of the gaps of J_+ and J_-. By assumption there exists $j_0 > l$ such that:

$$\text{for all } j \in J_+ \cap (j_0, +\infty),\ \text{for all } w \in R_l + j, \qquad \log|g(w)| \leq -\delta e^{j\mu_+}, \text{and}$$

$$\text{for all } j \in J_- \cap (-\infty, -j_0),\ \text{for all } w \in R_l + j, \qquad \log|g(w)| \leq -\delta e^{|j|\mu_-}.$$

On the other hand,

$$\text{for all } j \in J_+ \cap (j_0, +\infty),\ \text{for all } w \in R_l^- + j, \qquad |Re\ w| < j,$$

$$\text{for all } j \in J_- \cap (-\infty, -j_0),\ \text{for all } w \in R_l^+ + j, \qquad |Re\ w| < |j|.$$

As the rectangles $R_l^{\pm}$ are wider than the gaps of $J^{\pm}$, we get for all $w \in R_\infty$,

$$\text{if } Re\ w > j_0 \text{ then } \log|g(w)| \leq -\delta e^{\mu_+ Re\ w}, \text{ and}$$

$$\text{if } Re\ w < -j_0 \text{ then } \log|g(w)| \leq -\delta e^{\mu_- |Re\ w|}.$$

Define $h \in \mathcal{O}(S)$ by $h(w) = g(w)g(-w)$. Then $\log|h(w)| = \log|g(w)| + \log|g(-w)|$, so by the previous inequalities,

$$\text{for all } w \in R_\infty - R_{j_0},\ \log|h(w)| \leq -\delta e^{\mu|Re\ w|}.$$

In particular, $\sup_{\partial R_\infty} \log|h| = C_1 < +\infty$. Take any $\alpha > j_0$. Let ω be the harmonic function in the interior of the rectangle R_α with boundary values 0 on horizontal sides and 1 on vertical sides. One can find an explicit expression of ω as a series of functions, from which one can get a $C_2 > 0$ independent from α such that

$$\omega(0) \geq C_2 e^{-\frac{\pi\alpha}{h'}}.$$

On the boundary of R_α, the subharmonic function $w \mapsto \log|h(w)|$ is majorized by the harmonic function $w \mapsto C_1 - \delta e^{\mu\alpha}\omega(w)$. Therefore

$$\log|h(0)| \leq C_1 - \delta e^{\mu\alpha} C_2 e^{-\frac{\pi\alpha}{h'}} = C_1 - \delta C_2 e^{(\mu - \frac{\pi}{h'})\alpha}.$$

From $h' > \frac{\pi}{\mu}$ we get $\mu - \frac{\pi}{h'} > 0$. As the inequality is true for all $\alpha > j_0$, we conclude that $h(0) = 0$.

Now, for any ε so small that $R_\alpha + i\varepsilon \subset S$, we can repeat the above argument and obtain that $h(i\varepsilon) = 0$. By the principle of isolated zeros, h vanishes identically on S. Again by the principle of isolated zeros, it follows that g also vanishes identically. □

We denote

$$\mathbf{log}\,|\boldsymbol{z}| = \begin{bmatrix} \log|z_1| \\ \vdots \\ \log|z_d| \end{bmatrix} \in \mathbb{R}^d.$$

Then for all $\boldsymbol{k}$ and $\boldsymbol{z}$, a direct computation gives

$$\log|\boldsymbol{z}^{\boldsymbol{k}}| = <\boldsymbol{k}, \mathbf{log}\,|\boldsymbol{z}|> = k_1 \log|z_1| + \cdots + k_d \log|z_d| \ \in \mathbb{R}, \tag{5}$$

where $< \bullet, \bullet >$ denotes the usual inner product in $\mathbb{R}^d$.

For $p \geq 1$, denote $\mathbb{T} = (S^1)^p \subset \mathbb{C}^p$. For $\theta = (\theta_1, \ldots, \theta_p) \in \mathbb{T}$ and $j \in \mathbb{Z}$, denote $\theta^j = (\theta_1^j, \ldots, \theta_p^j)$. We will use the elementary

Lemma 2.2. *Let $\varepsilon > 0$. For any $\theta \in \mathbb{T}$, the set*

$$J = \{j \in \mathbb{N} \mid \quad \|(1,\dots,1) - \theta^j\|_\infty < \varepsilon\}$$

is infinite and has bounded gaps.

Proof. Consider the semigroup $X = \left\{ \theta^j \;\middle|\; j \in \mathbb{N} \right\} \subset \mathbb{T}$. Its closure $\bar{X}$ is a closed semigroup in a compact group, hence is a group.

If $\bar{X}$ is discrete then all θ_i's are roots of unity and the conclusion follows easily, so we can assume that $\bar{X}$ is a subtorus of $\mathbb{T}$ of dimension $n > 0$. For simplicity we assume that $n = 2$ (the case $n = 1$ is easier, and $n \geq 3$ is similar).

Denote $V = \mathrm{Lie}(\bar{X})$, which is a 2-dimensional subspace of $\mathrm{Lie}(\mathbb{T}) \approx \mathbb{R}^p$. Denote $U(\varepsilon)$ the ε-neighborhood of $(1,\dots,1)$ in $\mathbb{T}$. There exist a small neighborhood A of 0 in V and a small neighborhood B of $(1,\dots,1)$ in $\bar{X}$ such that $B \subset U(\varepsilon)$ and $exp : A \to B$ is a local group isomorphism. Take linear coordinates v_1, v_2 in V. We can assume that A is a ball for the associated sup-norm. Consider the open quadrants $Q_1,\dots,Q_4 \subset V$ determined by these coordinates.

As X is dense in B, we can find $j_1,\dots,j_4 \in \mathbb{N}$ such that each quadrant contains exactly one element of $\{\alpha_1 = exp^{-1}(\theta^{j_1}),\dots,\alpha_4 = exp^{-1}(\theta^{j_4})\}$. Now define a sequence $(a_p)_{p\in\mathbb{N}}$ in A by: $a_0 = \alpha_1$, and $a_{p+1} = a_p + \alpha_i$, where α_i is contained in the quadrant opposite to any closed quadrant containing a_p. Now the images by exp of this sequence give a subsequence of $(\theta^j)_{j\in\mathbb{N}}$ contained in $B \subset U(\varepsilon)$ showing that J is infinite with gaps not bigger than $\max\{j_1,\dots,j_4\}$. □

Denote $\mu = \log \rho(M)$. If $\mu = 0$ there is nothing to prove, so we assume $\mu > 0$. We choose a numbering of the eigenvalues of M such that

$$|\lambda_1| = \cdots = |\lambda_s| > |\lambda_{s+1}| \geq \cdots \geq |\lambda_{t-1}| > |\lambda_t| = \cdots = |\lambda_d|.$$

By our assumption on μ, $|\lambda_1| > 1 > |\lambda_d| > 0$. Denote $\mu_+ = \log|\lambda_1|$ and $\mu_- = -\log|\lambda_d|$. Then by definition of $\rho(M)$, $\mu = \max(\mu_+, \mu_-)$.

Lemma 2.3. *There exist $\tilde{\boldsymbol{z}} \in D$ and $\tilde{\boldsymbol{k}} \in \mathbb{Z}^d$ such that for some $\delta > 0$, $|\tilde{\boldsymbol{z}}^{j.\tilde{\boldsymbol{k}}}| > e^{\delta e^{j\mu_+}}$ for all $j \in J_+$, and $|\tilde{\boldsymbol{z}}^{j.\tilde{\boldsymbol{k}}}| > e^{\delta e^{|j|\mu_-}}$ for all $j \in J_-$, where $J_+ \subset \mathbb{Z}^{\geq 0}$ and $J_- \subset \mathbb{Z}^{\leq 0}$ are infinite with bounded gaps.*

Proof. Let $\{\boldsymbol{b_1},\dots,\boldsymbol{b_d}\} \subset \mathbb{C}^d$ be a Jordan basis for the linear action of M on $\mathbb{C}^d$. Up to exchanging some eigenvalues of equal modulus, we can assume that this numbering matches the above numbering of the eigenvalues.

Pick $\tilde{\boldsymbol{z}} \in D$, and denote $\mathbf{log}\,|\tilde{\boldsymbol{z}}| = \boldsymbol{u} = \sum_i u_i \boldsymbol{b_i} \in \mathbb{R}^d \subset \mathbb{C}^d$. As D is open we can assume that $u_1 \neq 0$ and $u_d \neq 0$.

When $j \to +\infty$, the dominant terms of $M^j \boldsymbol{u}$ come from the diagonals of the first blocks in the Jordan form of M. Namely we can write

$$M^j \boldsymbol{u} = |\lambda_1|^j \Big(\underbrace{\sum_{i=1}^{s} \theta_i^j u_i \boldsymbol{b_i}}_{\boldsymbol{u}'_{+,j}} + \boldsymbol{u}''_{+,j} \Big)$$

with $\boldsymbol{u}''_{+,j} \xrightarrow{j\to+\infty} 0$ and $\theta_i = \frac{\lambda_i}{|\lambda_1|} \in S^1,\ i = 1 \ldots s$.

And similarly

$$M^j \boldsymbol{u} = |\lambda_d|^j \Big(\underbrace{\sum_{i=t}^{d} \theta_i^j u_i \boldsymbol{b_i}}_{\boldsymbol{u}'_{-,j}} + \boldsymbol{u}''_{-,j} \Big)$$

with $\boldsymbol{u}''_{-,j} \xrightarrow{j\to-\infty} 0$ and $\theta_i = \frac{\lambda_i}{|\lambda_d|} \in S^1,\ i = t \ldots d$.

Denote $\boldsymbol{u}'_+ = \sum_{i=1}^{s} u_i \boldsymbol{b_i}$ and $\boldsymbol{u}'_- = \sum_{i=t}^{d} u_i \boldsymbol{b_i}$. Notice that $\boldsymbol{u}'_+$ and $\boldsymbol{u}'_-$ are linearly independent, and real because M and $\boldsymbol{u}$ are real. In particular, we can pick $\tilde{\boldsymbol{k}} \in \mathbb{Z}^d$ such that

$$< \boldsymbol{u}'_+, \tilde{\boldsymbol{k}} > > 3\delta \text{ and } < \boldsymbol{u}'_-, \tilde{\boldsymbol{k}} > > 3\delta$$

for some $\delta > 0$. By applying Lemma 2.2 to $\theta = (\theta_1, \ldots, \theta_s)$ for a small enough ε, we get an infinite $J_+ \subset \mathbb{Z}^{\geq 0}$ with bounded gaps, such that

$$\text{for all } j \in J_+,\ < \boldsymbol{u}'_{+,j}, \tilde{\boldsymbol{k}} > > 2\delta.$$

By applying Lemma 2.2 to $\theta = (\theta_t^{-1}, \ldots, \theta_d^{-1})$ for a small enough ε, we get an infinite $J_- \subset \mathbb{Z}^{\leq 0}$ with bounded gaps, such that

$$\text{for all } j \in J_-,\ < \boldsymbol{u}'_{-,j}, \tilde{\boldsymbol{k}} > > 2\delta.$$

Now, by (5) and (3), for all j, $\boldsymbol{k}$ and $\boldsymbol{z}$,

$$\log |\boldsymbol{z}^{j.\boldsymbol{k}}| =< \mathbf{log}\, |\boldsymbol{z}|, \boldsymbol{k} M^j >=< M^j(\mathbf{log}\, |\boldsymbol{z}|), \boldsymbol{k} > .$$

Thus for all $j \in J_+,\ \log |\tilde{\boldsymbol{z}}^{j.\tilde{\boldsymbol{k}}}| = |\lambda_1|^j (< \boldsymbol{u}'_{+,j}, \tilde{\boldsymbol{k}} > + < \boldsymbol{u}''_{+,j}, \tilde{\boldsymbol{k}} >)$, and as $\boldsymbol{u}''_{+,j} \xrightarrow{j\to+\infty} 0$, by discarding a finite number of (not big enough) elements from J_+, we get

$$\text{for all } j \in J_+,\ \log |\tilde{\boldsymbol{z}}^{j.\tilde{\boldsymbol{k}}}| > \delta |\lambda_1|^j = \delta e^{j\mu_+}.$$

Similarly, by discarding a finite number of elements from J_-, we get

$$\text{for all } j \in J_-,\ \log |\tilde{\boldsymbol{z}}^{j.\tilde{\boldsymbol{k}}}| > \delta |\lambda_d|^{-|j|} = \delta e^{|j|\mu_-}.$$

□

We now show by contradiction that $m\,\log\rho(M)\le 2\pi^2$: Assume that $m>\frac{2\pi^2}{\log\rho(M)}$. By (4), Proposition 2.1 and Lemma 2.3, it follows that $g_{\tilde{\boldsymbol{k}}}$ vanishes identically. But E_m is Stein, so any holomorphic function defined on some fiber (which is a closed submanifold) extends to a function on E_m. Equivalently, any function on $\{w\}\times D$ extends to a $\mathbb{Z}$-invariant function on $S_m\times D$. In particular, there exists such a function f satisfying $f(0,\boldsymbol{z})=\boldsymbol{z}^{\tilde{\boldsymbol{k}}}$. This function's Hartogs-Laurent expansion (2) has a non-zero coefficient corresponding to $\boldsymbol{z}^{\tilde{\boldsymbol{k}}}$. So $g_{\tilde{\boldsymbol{k}}}$ can not vanish identically.

2.2. **Proof of the "if" part of the Main Theorem.** We want to prove: If $m\,\log\rho(M)\le 2\pi^2$ then E_m is Stein.

By construction $D\hookrightarrow(\mathbb{C}^*)^d$ is $\mathbb{Z}$-equivariant, so $E_m(D,M)$ can be seen as subbundle of $E_m\big((\mathbb{C}^*)^d,M\big)$. In particular, $E_m(D,M)$ is a locally Stein open subset of $E_m\big((\mathbb{C}^*)^d,M\big)$. By the Docquier-Grauert theorem, it is therefore enough to prove the statement under the assumption that the fiber D is $(\mathbb{C}^*)^d$. We will simply write E_m instead of $E_m\big((\mathbb{C}^*)^d,M\big)$.

Outline of the proof: we roughly follow the line of argument of [Siu]. The key result here is Lemma 2.4, which plays the role of Siu's "Main Lemma". For this result the changes are of course essential because neither of the hypotheses made in [Siu] (vanishing of $b_1(D)$ and boundedness of D) holds here.

First we will assume that $m\log\rho(M)<2\pi^2$. We will show (in Lemma 2.4) that all monomials "$\boldsymbol{z}^{\boldsymbol{k}}$" can appear on the fibers of E_m, which is therefore holomorphically separable, and "fiberwise convex" with respect to plurisubharmonic (psh) functions (Lemma 2.5). It also follows that there exists a continuous strictly psh function ψ on E_m. Then we show that E_m is convex with respect to continuous psh functions (Lemma 2.6). From this and the existence of ψ it follows by [Nara] (*cf.* (0.2) in [Siu]) that if $\rho(M)=1$, or $\rho(M)>1$ and $m<\frac{2\pi^2}{\log\rho(M)}$, then E_m is Stein. For $\rho(M)=1$ we show the Steinness also when "$m=2\infty$", i.e., the base is $\mathbb{C}^*$.

The case of $\rho(M)>1$ and $m=\tilde{m}=\frac{2\pi^2}{\log\rho(M)}$ is then deduced as follows: From the above we obtain in particular an increasing sequence $(E^\nu)_{\nu\in\mathbb{N}}$ of Stein open subsets whose union is $E_{\tilde{m}}$ (e.g., $E^\nu=E_{\tilde{m}-1/\nu}$), and such that for all ν, E^ν is $\mathcal{O}(E^{\nu+1})$-convex. This implies that $E_{\tilde{m}}$ is Stein (*cf.* (1.2) in [Siu]).

Lemma 2.4. *Assume $m\log\rho(M)<2\pi^2$. Let $F=q\big(\{\tilde{w}\}\times(\mathbb{C}^*)^d\big)$ be any fiber of E_m. For all $\boldsymbol{k}\in\mathbb{Z}^d$, there exists a function f on E_m such that for all $\boldsymbol{z}\in(\mathbb{C}^*)^d$, $\quad f[\tilde{w},\boldsymbol{z}]=\boldsymbol{z}^{\boldsymbol{k}}$.*

Proof. Fix any $\boldsymbol{k}\in\mathbb{Z}^d$. Denote $\rho=\rho(M)$. We first assume that $\rho>1$. Pick $\varepsilon>0$ so small that

$$\big(\log(\rho+\varepsilon)\big)-\frac{2\pi^2}{m}<0. \tag{6}$$

Recall that $j.\boldsymbol{k} = M^j\boldsymbol{k}$. By a Jordan form argument, there exists $C > 0$ such that

$$\text{for all } j \in \mathbb{Z}, \quad \|j.\boldsymbol{k}\| \leq C(\rho + \varepsilon)^{|j|}.$$

Therefore for all j and $\boldsymbol{z}$,

$$|\boldsymbol{z}^{j.\boldsymbol{k}}| \overset{\text{by (5)}}{=} e^{<\log|z|, j.\boldsymbol{k}>} \leq e^{\alpha(\rho+\varepsilon)^{|j|}} \leq e^{\alpha e^{|j|\log(\rho+\varepsilon)}} \tag{7}$$

for some $\alpha > 0$ that depends on $\boldsymbol{z}$ but not on j.

Define Ω and Δ in $\mathcal{O}(S_m)$ by

$$\Omega(w) = e^{-2\cosh(\frac{2\pi^2}{m}w)} \text{ and } \Delta(w) = \frac{\sin \pi(w - \tilde{w})}{\pi(w - \tilde{w})}.$$

Key properties of these functions are:

$$|\Omega(w)| \sim e^{-e^{\frac{2\pi^2}{m}|Re\ w|}} \text{ when } w \in S_m \text{ and } |Re\ w| \to \infty, \tag{8}$$

$$\text{for all } w \in S_m, \quad \{\Delta(w+t) \mid t \in \mathbb{R}\} \text{ is bounded,} \tag{9}$$

$$\text{for all } j \in \mathbb{Z}, \quad \Delta(\tilde{w} + j) = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{if } j \neq 0. \end{cases} \tag{10}$$

Property (8) follows from the growth of the function $\cosh$ on any horizontal line contained in the strip $S = \{ w \in \mathbb{C} \mid Im\ |w| < \frac{\pi}{2} \}$, and that if $w \in S_m$ then $\frac{2\pi^2}{m} w \in S$. Property (9) follows from the periodicity of $\sin$.

Now define $f \in \mathcal{O}(E_m) = \mathcal{O}^{\mathbb{Z}}(S_m \times (\mathbb{C}^*)^d)$ by

$$f(w, \boldsymbol{z}) = \frac{1}{\Omega(\tilde{w})} \sum_{j \in \mathbb{Z}} \Omega(w+j)\ \Delta(w+j)\ \boldsymbol{z}^{j.\boldsymbol{k}}.$$

Let's check that f is well-defined: The $\mathbb{Z}$-invariance of the right hand side follows immediately from (3). Fix $(w, \boldsymbol{z})$. By (8), for some $C_1 > 0$, for all j,

$$|\Omega(w+j)| \leq C_1\, e^{-e^{\frac{2\pi^2}{m}|Re\ w+j|}}.$$

As $|Re\ w + j| \geq |j| - |Re\ w|$, by denoting $\beta = e^{-\frac{2\pi^2}{m}|Re\ w|} > 0$ we obtain

$$|\Omega(w+j)| \leq C_1\, e^{-\beta e^{\frac{2\pi^2}{m}|j|}}.$$

From this, (7) and (9), we obtain that for some $C_2 > 0$, for all j,

$$|\Omega(w+j)\ \Delta(w+j)\ \boldsymbol{z}^{j.\boldsymbol{k}}| \leq C_2\, e^{-\beta e^{\frac{2\pi^2}{m}|j|}} e^{\alpha e^{|j|\log(\rho+\varepsilon)}}.$$

The right hand side of the above inequality behaves like the *exponential* of

$$-\beta e^{\frac{2\pi^2}{m}|j|} \underbrace{\left(1 - \frac{\alpha}{\beta} e^{|j|\left(\log(\rho+\varepsilon) - \frac{2\pi^2}{m}\right)}\right)}_{=x_j}$$

It follows from (6) that $x_j \xrightarrow{|j|\to\infty} 1$. Thus the series is pointwise absolutely converging because its general term decays at a doubly exponential rate.

Moreover, as (8) and (9) hold locally uniformly with respect to w, so does the convergence of the series. Thus f is holomorphic in w. Besides, for a fixed w, the series is a (lacunary) Laurent series, so f is holomorphic in $\boldsymbol{z}$. By Hartogs's theorem, f is a holomorphic function on E_m.

It follows from (10) that for any $\boldsymbol{z} \in (\mathbb{C}^*)^d$, $f(\tilde{w}, \boldsymbol{z}) = \boldsymbol{z}^{\boldsymbol{k}}$.

Finally, if $\rho = 1$, the growth of $\|j.\boldsymbol{k}\|$ is now at most polynomial of some degree p'. Take an integer p such that $2p > p'$. We can apply a similar reasoning as above, with Ω defined by

$$\Omega(w) = e^{-w^{2p}},$$

which simultaneously fits the bill for any finite or infinite m. □

Corollary 2.5. *Assume $m \log \rho(M) < 2\pi^2$. Then*

1. *E_m is holomorphically separable,*
2. *there exists a continuous strictly psh function on E_m,*
3. *for any fiber F there exists a continuous psh function φ_F on E_m that restricts to an exhaustion on F.*

Proof. The pull-back to E_m of $\iota : w \mapsto w$ on the base A_m separates any two points that do not lie on the same fiber. Pick a fiber F. By Lemma 2.4 applied to F with $\boldsymbol{k}_1 = (1, 0, \dots, 0), \dots, \boldsymbol{k}_d = (0, \dots, 0, 1)$, we get functions $g_1, \dots, g_d$; and with $-\boldsymbol{k}_1, \dots, -\boldsymbol{k}_d$ we get $g_{d+1}, \dots, g_{2d}$. The corresponding map $G : E_m \to \mathbb{C}^{2d}$ restricts on F to a proper embedding because there is an isomorphism $H : (\mathbb{C}^*)^d \to F$ such that $GH(z_1, ..., z_d) = (z_1, ..., z_d, z_1^{-1}, ..., z_d^{-1})$. This shows in particular that functions on E_m separate points of F.

Now let φ_0 be any continuous psh exhaustion on $\mathbb{C}^{2d}$. Then $\varphi_F = \varphi_0 G$ is a continuous psh function on E_m that restricts to an exhaustion on F.

By the inverse mapping theorem, the functions $\iota, f_1, \dots, f_d$ give local coordinates on a neighborhood of any point of F. As F was arbitrary, functions on E_m give local coordinates around every point, so one can construct a continuous strictly psh function on E_m as in [Siu] (2.3) or [Dem] I (6.17). □

From now on, for the case of $\rho(M) > 1$, we look at all bundles E_m with $m < \tilde{m} = \frac{2\pi^2}{\log \rho(M)}$ as a family of open subsets of $E_{\tilde{m}}$.

Let $x = [w_1, \boldsymbol{z}_1] \in E_m$. Notice that if $v \in \mathbb{C}$ is so small that $w_1 + v$ belongs to S_m, then for any $(w_2, \boldsymbol{z}_2)$ such that $[w_2, \boldsymbol{z}_2] = [w_1, \boldsymbol{z}_1]$, ($w_2 + v$ also belongs to S_m and) $[w_1 + v, \boldsymbol{z}_1] = [w_2 + v, \boldsymbol{z}_2]$. For convenience, we state and prove the following lemma, which is a mere reformulation of an argument from [Siu].

Lemma 2.6. *Assume $m \log \rho(M) < 2\pi^2$. Then E_m is convex with respect to continuous psh functions, i.e., for any closed discrete sequence $\big(x_n = [w_n, \mathbf{z}_n]\big)_{n\in\mathbb{N}}$ there exists a continuous psh function φ that is unbounded on this sequence.*

Proof. If $p(x_n)$ has no accumulation point in the annulus A_m, then there is a function A_m whose pull-back on E_m gives the desired φ. So up to taking a subsequence, we can assume that $p(x_n)$ converges to $w \in A_m$, and thus $(\boldsymbol{z}_n)_{n\in\mathbb{N}}$ can not have any accumulation point in $D = (\mathbb{C}^*)^d$.

If $\rho(M) > 1$, take m' such that $m < m' < \frac{2\pi^2}{\log \rho(M)}$, and $\varepsilon > 0$ such that: If $w \in S_m$ and $|v| \leq \varepsilon$, then $w + v \in S_{m'}$. If $\rho(M) = 1$, we assume that m is infinite (i.e., $S_m = \mathbb{C}$) and take any $\varepsilon > 0$.

Denote $F = p^{-1}(w)$. Let φ_F be the continuous psh function on $E_{m'}$ obtained from Corollary 2.5 (3). Define a continuous psh function φ on E_m by

$$\varphi[w, \boldsymbol{z}] = \sup_{|v|\leq\varepsilon} \varphi_F[w + v, \boldsymbol{z}].$$

For n big enough, $|w - w_n| < \varepsilon$, so $\varphi(x_n) \geq \varphi_F[w, \boldsymbol{z}_n]$. Therefore φ satisfies to the required properties. □

The Main Theorem is now proved.

2.3. **Proof of Theorem 2.** The proof is simply a refinement of that of Sect. 2.1, based on the extra assumptions we made. We omit the details.

Let $f \in \mathcal{O}(E_m)$. Let $\boldsymbol{k}$ be any nonzero element of $\mathbb{Z}^d$. Take λ_+ and λ_- eigenvalues of M with respectively maximal and minimal modulus. Denote $\mu_+ = \log|\lambda_+|$ and $\mu_- = -\log|\lambda_-|$.

Take $\boldsymbol{v} \in \mathbb{C}^d$ an eigenvector (for the linear action of M on $\mathbb{C}^d$) associated to λ_+. We choose $\boldsymbol{v} \in \mathbb{R}^d$ in case λ_+ is real. Define $W = Span_{\mathbb{R}}\{Re\, \boldsymbol{v}, Im\, \boldsymbol{v}\}$. Then W is an M-invariant subspace of $\mathbb{R}^d$ on which M acts by a rotation-dilation of factor λ_+ (if $\lambda_+ \in \mathbb{R}$ then $\dim W = 1$ and we have a dilation of factor λ_+). As the characteristic polynomial of M is irreducible, there exists $\boldsymbol{u} \in W$ such that $<\boldsymbol{u}, \boldsymbol{k}> > 0$ (*cf.* [Fann-Wol] Lemma 3 (f)).

Take $\boldsymbol{z}_+ \in (\mathbb{C}^*)^d$ such that $\log|\boldsymbol{z}_+| = \boldsymbol{u}$. Now by a similar (but simpler) argument than for Lemma 2.3, we get an infinite $J_+ \subset \mathbb{N}$ with bounded gaps such that $|\boldsymbol{z}_+^{j.\boldsymbol{k}}| > e^{\delta e^{j\mu_+}}$ for all $j \in J_+$.

Similarly, there exists $\boldsymbol{z}_- \in (\mathbb{C}^*)^d$ and an infinite $J_- \subset -\mathbb{N}$ with bounded gaps such that $|\boldsymbol{z}_-^{j.\boldsymbol{k}}| > e^{\delta e^{|j|\mu_-}}$ for all $j \in J_-$.

Let $g_{\boldsymbol{k}}$ be the coefficient of f in the Hartogs-Laurent series (2). Then, as in Sect. 2.1, we obtain that $g_{\boldsymbol{k}}$ vanishes identically. We conclude that $f(w, \boldsymbol{z}) = g_{\boldsymbol{0}}(w)$. □

3. Open problems.

3.1. **Serre problem for bounded domains.** Before Cœuré and Lœb's counterexample, it was conjectured by Siu that any bounded domain belonged to $\mathcal{S}$ (*cf.* 1.1). So far, the only known counterexamples are (equivariant subsets of) Reinhardt domains that generalize Cœuré and Lœb's. This raises the problems, already open in dimension two, of the existence of other counterexamples, and of giving a characterization of all bounded domains not in $\mathcal{S}$ (*cf.* [Ch-Zh]).

3.2. **Siu's conjecture.** For any of the known counterexamples to that conjecture, our results say that, for a given transition automorphism, a bundle with fiber D over a sufficiently thin annulus is Stein. Note also that for another bundle, with fiber $\mathbb{C}^2$, an analogous result is proved in [Dem1]. Therefore Siu's conjecture is slightly rekindled, and this begs the question: Does this interplay between fiber and base, that gives a Stein total space, also happen for another "sufficiently thin" Stein base? For other domains not in $\mathcal{S}$ (if they exist)?

3.3. **Stein and paralellizable manifolds.** It is conjectured that a Stein and parallelizable n-manifold (i.e., with trivial tangent bundle) can be realized as a Riemann domain over $\mathbb{C}^n$ (*cf.* the nice survey [For]).

The bundles $E_m(D, M)$ are parallelizable, and when m is so big that $E = E_m(D, M)$ is not Stein, then E is not a Riemann domain (*cf.* [Siu]). So by proving that for small m, E is a Riemann domain, one would obtain a good case study for the conjecture: a continuous family of parallelizable manifolds (with a description of functions on each of them) that degenerate from Stein and Riemann domains to non-Stein and non-Riemann domains.

Acknowledgements. I am indebted to Al Baernstein and Leonid Kovalev for showing me that a stronger version of Corollary 6.4 in [Zaf] followed from a "classical" Phragmén-Lindelöf-type argument. A variation of their statement yields an important ingredient in Sect. 2.1. I am very grateful to Karl Oeljeklaus for many helpful conversations, and to Charles Li for his listening and help. Thanks also go to Alan Huckleberry for asking me a question that is answered by Theorem 2.

References

[Baer-Kov] **A. Baernstein, L. Kovalev:** Private communication.

[Ch-Zh] **Chen, B.-Y., Zhang, J.-H.:** "The Serre problem on certain bounded domains". Asian J. Math. 7 (2003), no. 4, 511–518.

[Cœ-Lœ] **G. Cœuré, J.-J. Lœb:** "A counterexample to the Serre problem with a bounded domain of $\mathbb{C}^2$ as fiber". Ann. of Math., 122 (1985), 329–334.

[Dem] **J.-P. Demailly:** "Complex analytic and algebraic geometry". http://www-fourier.ujf-grenoble.fr/~demailly/books.html

[Dem1] **J.-P. Demailly:** "Un exemple de fibré holomorphe non de Stein à fibre $\mathbb{C}^2$ ayant pour base le disque ou le plan". Invent. Math. 48 (1978), no. 3, 293–302.

[Die-For] **K. Diederich, J. Fornaess:** "Pseudoconvex domains: bounded strictly plurisubharmonic exhaustion functions". Inventiones Math. 39, 129–141 (1977).

[For] **F. Forstnerič:** "The homotopy principle in complex analysis: a survey". Explorations in complex and Riemannian geometry, 73–99, Contemp. Math., 332, Amer. Math. Soc., Providence, RI, 2003.

[Fann-Wol] **A. Fannjiang, L. Wołowski:** "Lech Noise induced dissipation in Lebesgue-measure preserving maps on d-dimensional torus". J. Statist. Phys. 113 (2003), no. 1-2, 335–378.

[Mok] **N. Mok:** "Le problème de Serre pour les surfaces de Riemann". C. R. Acad. Sci. Paris Sér. A-B 290 (1980), no. 4, A179–A180.

[Nara] **R. Narasimhan:** The Levi problem for complex spaces. II. Math. Ann. 146 (1962), 195–216.
[Oel-Zaf] **K. Oeljeklaus, D. Zaffran:** "Steinness of bundles with fiber a Reinhardt bounded domain". Bull. Soc. Math. France 134 (2006), no. 4, 451–473.
[Pfl-Zwo] **P. Pflug, W. Zwonek:** "The Serre problem with Reinhardt fibers". Ann. Inst. Fourier 54 (2004), no. 1, 129-146.
[Roy] **H.L. Royden:** "Holomorphic fiber bundles with hyperbolic fiber". Proc. Amer. Math. Soc. 43 (1974), 311–312.
[S-Z] **A. Schinzel, H. Zassenhaus:** "A refinement of two theorems of Kronecker". Mich. Math. J. 12 (1965), 81-85.
[Shim] **S. Shimizu:** "Automorphisms of bounded Reinhardt domains". Japan. J. Math. (N.S.) 15 (1989), no. 2, 385–414.
[Siu] **Y.T. Siu:** "Holomorphic fiber bundles whose fibers are bounded Stein domains with zero first Betti number". Math. Ann. 219 (1976), no. 2, 171–192.
[Sko] **H. Skoda:** "Fibrés holomorphes à base et à fibre de Stein". Invent. Math. 43 (1977), no. 2, 97–107.
[Smy] **C. J. Smyth:** "The Mahler measure of algebraic numbers: a survey". http://arxiv.org/abs/math/0701397
[Steh] **J.-L. Stehlé:** "Fonctions plurisousharmoniques et convexité holomorphe de certains fibrés analytiques". C. R. Acad. Sci. Paris Sér. A 279 (1974), 235–238.
[Tit] **E.C. Titchmarsh:** "The Theory of Functions", 2nd ed, Oxford Univ. Press, Oxford, 1939.
[Vou] **P. Voutier:** "An effective lower bound for the height of algebraic numbers". Acta Arith. 74 (1996), 8195.
[Zaf] **D. Zaffran:** "Serre problem and Inoue-Hirzebruch surfaces". Math. Ann. 319 (2001), no. 2, 395–420.